Stationary Random Processes and Long-Run Behaviors

Mathias Winther Madsen
mathias.winther@gmail.com

Institute for Logic, Language, and Computation
University of Amsterdam

18 November 2015

The Ergodic Theorem

The Ergodic Theorem

Ergodicity implies the uniqueness of limiting frequencies; stationarity implies existence.

Decomposition by Partitioning

Partitioning

The relation "has the same limiting frequencies as" is an equivalence relation on the space of ergodic random processes. Each equivalence class contains at most one stationary random process.

(*Proof*: By the extension theorem.)

Decomposition by Partitioning

Partitioning

A mixture of stationary and ergodic distributions is stationary but not necessarily ergodic. A non-ergodic distribution can be split up into a mixture of ergodic distributions.

(*Proof*: By partitioning.)

Non-Ergodic Processes are Mixtures

Indefinite Repetition

Pick a letter at random and print it forever:

`AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA ...`

Half-Deterministic: $\frac{1}{2}$Bernoulli(0) + $\frac{1}{2}$Bernoulli(1/2)

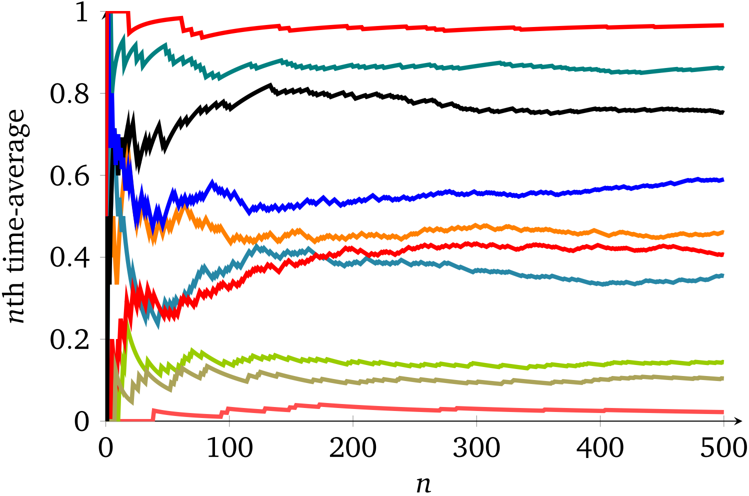
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$$
$$1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, \ldots$$

Beta Urn

Repeatedly draw a marble from an urn with 1 blue and 1 red marble and add an identical marble to the urn:

`R B R B R B R R B R B B R R R R B R B B B R R R B ...`

Non-Ergodic Processes are Mixtures



Attractor Identification

Definition

A distribution P is **absolutely continuous** with respect to a reference distribution P^* if

$$P^*(B) = 0 \qquad \Longrightarrow \qquad P(B) = 0$$

Attractor Processes

If a random process P is absolutely continuous with respect to a stationary and ergodic process P^*, then their limiting time-averages coincide.

(*Proof*: $P^*(f^*) = 1$, so $P(f^*) = 1$ by absolute continuity.)

Attractor Identification

Fixed-Length Repetitions

Repeatedly pick a letter at random and print it three times:

`LLL EEE HHH QQQ MMM QQQ OOO TTT EEE YYY XXX GGG ...`

$X_n \sim \{1, 2, 3, \ldots, 2^n\}$

$$1,\ 2,\ 3,\ 6,\ 14,\ 20,\ 39,\ 50,\ 245,\ 132,\ 563,\ 194,\ 3813,\ \ldots$$

Ups and Downs

Repeatedly print $k \sim \text{Geometric}(1/2)$ left-parentheses and immediately after, k right-parentheses:

$$()\,((()))\,((()))\,(())\,(())\,((()))\,(())\ \ldots$$

Attractor Identification

$P = \frac{1}{2}\text{Bernoulli}(1/3) + \frac{1}{2}\text{Bernoulli}(2/3)$

$$1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 1, \ldots$$

Beta Urn

Repeatedly draw a marble from an urn with 1 blue and 1 red marble and add an identical marble to the urn:

`R B R B R B R R B R B B R R R B R B B B R R R B ...`